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GRADED CONTRACTIONS OF VIRASORO ALGEBRAS

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Abstract

We describe graded contractions of Virasoro algebra. The highest weight representations of Virasoro algebra are constructed. The reducibility of representations is analysed. In contrast to standart representations the contracted ones are reducible except some special cases. Moreover we find an exotic module with null-plane on fifth level.

1 Introduction

The Virasoro algebra takes an important part in strings theory, integrable models and two-dimensional conformal field theory. Resently it has been investigated the extensions of Virasoro algebra leading to W-algebras, which contain the Virasoro algebra as a subalgebra. W-algebras are appeared in integrable models [1, 2] and then in two-dimensional conformal field theory [3]. Thereafter the notions of W-strings, W-gravitation have arisen and W-models of conformal field theory have been created [4]. Hence the extensions of symmetries on the base of the Virasoro algebra are very fruitful.

A different way of extension of Virasoro algebra is a contraction. In this case new nonsemisimple algebras containing the Virasoro algebra as a subalgebra are arisen. Contractions of affine Kac-Moody algebras were studied in [5, 6] and similar effect was obtained [7]. Nonsemisimple algebras are used as algebras of internal symmetries of different physical models: gauge theories [8], WZNW models [9, 10]. Sugavara construction was developed for nonsemisimple algebras [11]. The purpose of this paper is the investigation of contractions of Virasoro algebra and its representations with the help of the graded contraction method [7, 12].

This paper is organized as follows. In the second section we briefly present the main ideas of graded contraction method. In Sec.3 the grading of Virasoro algebra by cyclic groups $\mathbb{Z}_2$ is suggested. In Sec.4 we recall the base facts of reducible theory of modules over Virasoro algebra and describe their $\mathbb{Z}_2$-grading. Sec.5 contains the results of some contractions of Virasoro algebra. In Sec.6 contractions of modules and some problems of their reducibility are investigated.

2 Graded contractions of Lie algebras and their representations

A Lie groups (algebras) contraction makes possible to get new Lie groups (algebras) from some initial ones. Moreover one can construct a representation of contracted algebras from a representation of starting algebra. The graded contractions [6, 12] are defined in such a way that the grading of both Lie algebra and its representation are preserved.

First we recall some definitions. Let a Lie algebra L is graduated by an abelian group G as follows

$$L = \bigoplus L_j, \quad [L_j, L_i] \subseteq L_{j+i}, \quad j, i \in G. \tag{1}$$

A Lie algebra L^ε is called a G-graded contraction of the algebra L if L^ε is isomorphic to L as vector space, L^ε has grading (1) and new commutation relations are

$$[L^\varepsilon_j, L^\varepsilon_k]_\varepsilon := \varepsilon_{jk}[L_j, L_k] \subseteq \varepsilon_{jk} L^\varepsilon_{j+k}, \tag{2}$$

where the matrix ε is a solution of the equations

$$\varepsilon_{jk}\varepsilon_{m,j+k} = \varepsilon_{km}\varepsilon_{j,m+k} = \varepsilon_{mj}\varepsilon_{k,m+j}, \quad \varepsilon_{jk} = \varepsilon_{kj}. \tag{3}$$

The first set of equations in (3) is the consequence of Jacoby identity and the second one follows from antisimmetry of commutator.

G-grading of L-module V is obtained from decomposition (1) and looks as follows

$$V = \bigoplus V_m, \quad L_j V_m \subseteq V_{j+m}, \tag{4}$$

$$[L_j, L_k]V_m = L_j L_k V_m - L_k L_j V_m \subseteq V_{j+k+m}. \tag{5}$$

A G-graded contraction of algebra representation is defined by

$$L_j^{\varepsilon} V_m^{\varepsilon,\psi} := \psi_{jm} L_j V_m \subseteq \psi_{jm} V_{j+m}^{\varepsilon,\psi}. \tag{6}$$

where the matrix ψ satisfy the following equations [7]

$$\varepsilon_{jk}\psi_{j+k,m} = \psi_{km}\psi_{j,k+m} = \psi_{jm}\psi_{k,j+m}. \tag{7}$$

Hereafter we shall use for grading only cyclic group $G = \mathbb{Z}_2$. It is convenient to write down the vector space V and action of algebra L for $\mathbb{Z}_2$-grading in the form

$$V = \begin{pmatrix} V_0 \\ V_1 \end{pmatrix}, \quad LV = \begin{pmatrix} L_0 & L_1 \\ L_1 & L_0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} L_0 V_0 + L_1 V_1 \\ L_1 V_0 + L_0 V_1 \end{pmatrix}, \tag{8}$$

explicitly showing the structure of grading. Then we have for contracted module

$$L^{\psi} V = \begin{pmatrix} L_0 & L_1 \\ L_1 & L_0 \end{pmatrix}^{\psi} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} \psi_{00} L_0 & \psi_{11} L_1 \\ \psi_{10} L_1 & \psi_{01} L_0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix}. \tag{9}$$

3 Grading of Virasoro algebra

The Virasoro algebra $\mathcal{L}(\mathrm{c})$ is the central extension of the algebra of vector fields on the circle with the basis c, l_n ($n \in \mathbb{Z}$) and the commutation relations

$$[l_n, l_m] = (n-m) l_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{10}$$

where c is the central charge commuting with all generators. This algebra has the natural $\mathbb{Z}$-grading: deg $l_k = k$, deg $c = 0$ and may be graded also by cyclic groups $\mathbb{Z}_n$. The method of $\mathbb{Z}_2$-grading of the Virasoro algebra consists in follows. The set of the Virasoro algebra generators is divided on even $L_0 = \{A_n, c\}$ and odd $L_1 = \{B_n\}$ parts, where

$$A_n = \frac{1}{2}\left(l_{2n} + \frac{c}{8}\delta_{n,0}\right), \quad B_n = \frac{1}{2}l_{2n+1}, \tag{11}$$

then the Z_2-grading conditions are held

$$\mathcal{L} = L_0 \bigoplus L_1, \quad [L_0, L_0] \subseteq L_0, \quad [L_0, L_1] \subseteq L_1, \quad [L_1, L_1] \subseteq L_0.$$

and the commutation relations of Virasoro algebra for new generators look as

$$\begin{aligned}
[A_n, A_m] &= (n-m)A_{n+m} + \frac{2c}{12}(n^3 - n)\delta_{n+m,0}, \\
[B_n, B_m] &= (n-m)A_{n+m+1} + \frac{2c}{12}(n - \frac{1}{2})(n + \frac{1}{2})(n + \frac{3}{2})\delta_{n+m+1,0}, \\
[A_n, B_m] &= (n - m - \frac{1}{2})B_{n+m}.
\end{aligned} \tag{12}$$

The first line of equations means that the subalgebra $\{A_n, c\}$ is again Virasoro algebra but with central charge 2c. In the case of $\mathbb{Z}_n-$grading we have Virasoro subalgebra with central charge nc. This looks like the result of generating the orbifold algebras by $\mathbb{Z}_\lambda$-orbifold induction procedure $c \to \lambda c$ [13].

4 Reducibility of representations. $\mathbb{Z}_2$-grading

In this section we recall a basic notions and results of the theory of Virasoro algebra representations [14] and describe their $\mathbb{Z}_2$-grading. Let M be the module over $\mathcal{L}(c)$ and there is the vector $|v\rangle$ such that

$$l_0|v\rangle = h|v\rangle, \quad \hat{c}|v\rangle = c|v\rangle, \quad l_k|v\rangle = 0, \ k > 0, \tag{13}$$

then $|v\rangle$ is a highest vector. The space spanned by vectors $l_{-i_1} \ldots l_{-i_m}|v\rangle$, where $i_1 \geq \ldots \geq i_m > 0$, form a highest vector representation $M(h, c)$. All vectors of module are classified by levels. The level number is given by $n = i_1 + \ldots + i_m$. For example $l_{-2}l_{-1}| \ v \ \rangle$ belongs to the third level. Therefore the module has natural $\mathbb{Z}$-grading $M(h, c) = \bigoplus M^n(h, c)$. The number of basis vectors on the n-th level is defined by the number $p(n)$ of partitions of n on positive integer numbers: $\dim M^n(h,c) = p(n)$. Some values of $p(n)$ are given below:

$$p(0) = 1,\ p(1) = 1,\ p(2) = 2,\ p(3) = 3,\ p(4) = 5,\ p(5) = 7.$$

The investigation of reducibility conditions of module is an important problem. A module is called degenerate if there is a null-vector $|\chi_n\rangle$, such that

$$l_0|\chi_n\rangle = (h+n)|\chi_n\rangle, \quad l_k|\chi_n\rangle = 0, \quad k > 0,$$

where n is called the degeneracy of level. The degenerated module contains the submodule with highest vectors $|\chi_n\rangle$ on the n-th level and therefore is a reducible module. Null-vector is found as a linear combination of basis vectors with unknown coefficients. The conditions of null-vectors lead to the system of linear equations with two parameters h and c. The number of equations and the number of unknowns are given by $N_1(n) = p(n-1) + p(n-2)$ and $N_2(n) = p(n) - 1$ respectively.

On the second level we have two equations and only one unknown. The relation between h and c on this level is given by

$$h = \frac{5 - c \pm \sqrt{(c-25)(c-1)}}{16}.$$

The number of equations on the next levels increases faster then the number of unknowns and therefore the existence of null-vectors becomes a rarity. The problem of searching of null-vectors is completely solved with the help of bilinear form on the $M(h,c)$. Let w be an antiautomorphizm $\mathcal{L}(c)$ defined by $w(l_i) = l_{-i}$, $w(c) = c$, then there is a symmetrical bilinear form $(\ |\ )$, such that $(a|lb) = (w(l)a|b)$, where $a, b \in M(h,c),\ l \in \mathcal{L}(c)$. Let the $p(n) \times p(n)$ matrix $K^n(h,c)$ is the matrix of bilinear form of basis vectors on n-th level. The vanishing of determinant of matrix $K^n(h,c)$ is the condition of existence of null-vector [14]. There are some possibilities depending on the values of parameters h, c: 1) the module is irreducible, then there are no null-vectors; 2) the submodules generated by null-vectors are embedded one into another; 3) there are two submodules containing all other submodules. In the last case the modules $M(h,c)$ are defined by

$$c = 1 - \frac{6(q-p)^2}{pq}, \qquad h = \frac{(qr-ps)^2 - (q-p)^2}{4pq},$$

where $0 < r < p$, $0 < s < q$, for integers r, s and $q = p + 1 = 3, 4, \ldots$. The last modules describe the space of fields of minimal models in conformal fields theory. Degeneracy of module leads to the differential equations on the fields correlators [15].

We are coming now to the describing of $\mathbb{Z}_2$-grading of module $M(h, c)$. We shall consider that the Virasoro algebra is $\mathbb{Z}_2$-graded by the above described method. The representation space may be divided on two subspaces

$$M(h, c) = M_0(h, c) + M_1(h, c),$$

$$M_0(h, c) = \bigoplus M^{2n}(h, c), \quad M_1(h, c) = \bigoplus M^{2n+1}(h, c).$$

The grading of elements $l_{-i_1} \ldots l_{-i_m}$ of enveloping algebra is realized by decomposition on two subset with odd and even value of sum $i_1 + \ldots + i_m$. The vectors from $M_0(h, c)$ and $M_1(h, c)$ in terms of generators A and B look as

$$AA \ldots AA \underbrace{BB \ldots BB}_{k} |v\rangle, \tag{14}$$

where k is even for $M_0(h, c)$ and odd for $M_1(h, c)$. Then the following equations of $\mathbb{Z}_2$-grading are realized

$$AM_0 \subseteq M_0, \ AM_1 \subseteq M_1, \ BM_0 \subseteq M_1, \ BM_1 \subseteq M_0.$$

The structure of the $\mathbb{Z}_2$-grading module may be written as

$$LM = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \end{pmatrix} = \begin{pmatrix} AM_0 + BM_1 \\ BM_0 + AM_1 \end{pmatrix}. \tag{15}$$

Let us note that antiautomorphism w in $\mathbb{Z}_2$-graded notation looks as $w(A_i) = A_{-i}, \ w(B) = B_{-i+1}, \ w(c) = c.$

5 $\mathbb{Z}_2$-graded contraction of Virasoro algebra

Consider the contraction given by solution $\varepsilon^\alpha = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ of equations (3). The commutation relations (12) take the form

$$[A_n, A_m] \ = \ (n - m)A_{n+m} + \frac{2c}{12}(n^3 - n)\delta_{n+m,0},$$

$$[A_n, B_m] = (n - m - \frac{1}{2})B_{n+m}, \qquad [B_n, B_m] = 0.$$

The contracted algebra has the structure of *semidirect* sum of Virasoro algebra with double central charge $\mathcal{L}(2c) = \{A_n, 2c\}$ and infinite dimensional abelian one $\{B_n\}$. It may be called as inhomogeneous Virasoro algebra. The second solution of equations (3) $\varepsilon^\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ leads to the contracted Virasoro algebra, which is a *direct* sum of $\mathcal{L}(2c)$ and abelian $\{B_n\}$

$$\begin{aligned} [A_n, A_m] &= (n - m)A_{n+m} + \frac{2c}{12}n(n^2 - 1)\delta_{n+m,0}, \\ [A_n, B_m] &= 0, \qquad [B_n, B_m] = 0. \end{aligned}$$

6 $\mathbb{Z}_2$-Grading contractions of Virasoro modules

It is easily to understand the structure of contracted $\mathbb{Z}_2$-module using the relations (15) and (9). We take the solution of equations (7) $\psi^\alpha = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. In this case

$$LM = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \end{pmatrix} = \begin{pmatrix} AM_0 \\ BM_0 + AM_1 \end{pmatrix}. \tag{16}$$

Therefore all properties of contracted module may be obtained by assuming

$$BM_1 = 0. \tag{17}$$

which means that all vectors of the form (14) with $k \geq 2$ are vanish. In other words the contracted module is obtained from the initial one by factorization with respect to condition $BB|v\rangle = 0$.

It follows from (11) that the parameters of contracted $M_\psi(h, c)$ and initial $M(h_0, c_0)$ modules are connected by relations $c = 2c_0,\ h = \frac{1}{2}(h_0 + \frac{c_0}{8})$. It is easily to understand that the structure of vectors on the even level $2n$ of contracted module and the structure of vectors on the level n of initial one are identical. Therefore $\dim M_\psi^{2n}(h, c) = \dim M^n(h, c) = p(n)$. The vectors of odd level look as $A_{-i_1} \ldots A_{-i_n} B_{-q}|v\rangle$. The dimension of odd level is given by

$$\dim M_\psi^{2n+1}(h, c) = \sum_{k=0}^{n} p(k). \tag{18}$$

Phisical requirement that the spectrum must be bounded from below leads to the highest vector condition (13). Since any element l_k is expressed as multiple commutators of l_1 and l_2, then in (14) instead of $l_k|v\rangle = 0,\ k > 0$ it is enough to impose two conditions: $l_1|v\rangle = 0,\ l_2|v\rangle = 0$. Then the highest vector conditions (13) are simplified

$$A_0|v\rangle = h|v\rangle,\ \hat{c}|v\rangle = c|v\rangle,\ B_0|v\rangle = 0,\ A_1|v\rangle = 0. \tag{19}$$

For ε^{α}-contracted Virasoro algebra these conditions needs to be modified by adding the equation $A_2|v\rangle = 0$, since otherwise the generators $A_k,\ k > 1$ are not generated by commutators of B_0 and A_1. Thus the existence conditions of null-vector on the level n are

$$A_0|\chi\rangle = (h+n)|\chi\rangle,\ B_0|\chi\rangle = 0,\ A_1|\chi\rangle = 0,\ A_2|\chi\rangle = 0. \tag{20}$$

The increase of number of equations on even levels with compared to (19) leads to vanishing of the null-vectors on these levels. It is possible to verify this fact by direct calculations. The relations $B_0|\chi\rangle = 0$ for odd level are held due to (17). Thus the number of conditions remains the same therefore one would expect the presence of null-vectors on odd levels. The number N_1 of equations and the number N_2 of unknown coefficients, which defined the existence of the null-vectors on $(2n+1)$-th level are easily obtained from (18) and (20)

$$N_1 = \sum_{k=0}^{n-1} p(k) + \sum_{k=0}^{n-2} p(k), \qquad N_2 = \sum_{k=0}^{n} p(k) - 1$$

It follows from this equations that $N_1 = N_2$ up to eleventh level and $N_1 > N_2$ for higher-order levels.

Let us investigate some first levels in details. One find by direct calculations that there are null-vectors on the first and on the third levels

$$|\chi\rangle_1 = B_{-1}|v\rangle, \qquad |\chi\rangle_3 = \left(B_{-2} - \tfrac{5}{4h+2}A_{-1}B_{-1}\right)|v\rangle,$$

for any values of h, c. On the fifth level the null-vector looks as

$$|\chi\rangle_5 = \left(B_{-3} + \alpha A_{-2}B_{-1} + \beta A_{-1}B_{-2} + \gamma A_{-1}^2 B_{-1}\right)|v\rangle,$$

where α, β, γ are solutions of system with parameters h and c

$$\begin{aligned} (4h+6)\beta \qquad\qquad\qquad &= 7, \\ (4+8h+c)\alpha + 15\beta + (12h+6)\gamma &= -9, \\ 6\alpha + 5\beta + (8h+8)\gamma &= 0. \end{aligned}$$

It is immediately follows from the first equation that there are no null-vectors on the straight line $h = -3/2$. The substitution of the first equation into the others reduces the analysis of system to possible disposition of a straight lines on the plane (α, γ). Let $D(h, c)$ be the determinant of matrix composed from the coefficients of the left part of system and let $D_i(h, c)$, $i = 1, 2$ be the determinant of the matrix, which is obtained from the previous one by the replacement of i-th column with the column of the right part of the system. There are three possibility depending on the parameters h and c: 1) the straight lines are crossed – in this case there are null-vectors; 2) the straight lines are parallel – in this case there are not null-vectors; 3) the straight lines are coincided – there are null-vectors, moreover the null-vectors become two-dimensional or "null-plane". The second case is realized on the straight line $h = -\frac{3}{2}$, and two curves $D(h, c) = 0$ on (h,c)-plane, i.e.

$$h = \frac{-3 - c \pm \sqrt{(c-25)(c-1)}}{16},$$

The third case is a possibile at two points on (h,c)-plane: $A(h_1 = -\frac{3}{2}, c_1 = 26)$ and $B(h_2 = \frac{11}{24}, c_2 = -\frac{184}{105})$ which are the intersection of the curve $D(h, c) = 0$ and the straight line $D_2(h, c) = 0$, i.e. $c = (176 - 496h)/35$. However the point A is on the straight line $h = -\frac{3}{2}$, where there are not null-vectors. So the null-plane on the fifth level appears at the unique values of parameters corresponding to the point $B\,(h_2, c_2)$. Similar result was obtained in paper [16] where it was shown that certain Verma modules over the $N = 1$ Ramond algebra contain degenerate two-dimensional singular vector spaces. For the rest values of parameters h and c the first case is realized.

On the seventh and ninth levels the numbers of equations and unknown coefficients are coincided and are equal to 6 and 11 respectively. However the calculations become too cumbersome, although it is clear that null-vectors are absent only in a special cases. Starting with eleventh level the number of equations begins to exceed the number of unknowns $N_1(11) = 19,\ N_2(11) = 18$ and the situation becomes apparently similar to the case of noncontracted module. The attempt of using the ordinary bilinear form as in the case of noncontracted algebra does not leads to the solution of the problem of searching of null-vectors. The point is that all vectors on odd levels are orthogonal to each other due to (17). It is an ordinary thing for a finite Lie algebra that the Killing form is degenerated under contraction. At this point it may be useful the results of the work [17], where the way of construction of nondegenerate forms for the contracted algebras was proposed.

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